Farkas Lemma and proof of duality

Source: Chapter 4 of Matoušek

Farkas Lemma: Let $A \in \mathbb{R}^{m\times n}$ and $\mathbf{b} \in \mathbb{R}^m$. Exactly one of the following holds

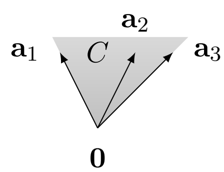
- $\exists \mathbf{x} \in \mathbb{R}^n$ such that $A\mathbf{x} = \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$
- $\exists \mathbf{y} \in \mathbb{R}^m$ such that $\mathbf{y}^T A \geq \mathbf{0}^T$ and $\mathbf{y}^T\mathbf{b} < 0$

1: Is it possible to satisfy both conditions at the same time? Why?

Solution: No. Suppose for contradiction that both are satisfied at the same time. This gives $\mathbf{y}^T A\mathbf{x} = \mathbf{y}^T\mathbf{b}$. The left-hand side is ≥ 0 while the right-hand side is negative.

A (convex) **cone** is a set $C \in \mathbb{R}^d$ for which $\mathbf{x}, \mathbf{y} \in C$ and $a, b \geq 0$ implies $a\mathbf{x} + b\mathbf{y} \in C$.
A cone C generated by $X = \{\mathbf{a}_1, \ldots, \mathbf{a}_n\} \subseteq \mathbb{R}^d$ are all linear combinations of vectors in X with nonnegative coefficients. That is

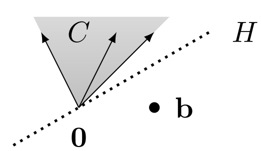
$$C = \{t_1\mathbf{a}_1 + t_2\mathbf{a}_2 + \cdots + t_n\mathbf{a}_n : t_i \geq 0\} \subseteq \mathbb{R}^d.$$



A convex cone can be defined for any generating set X. If X is finite, then C is closed.

Geometric version of Farkas Lemma: Let $\mathbf{a}_1, \ldots, \mathbf{a}_n, \mathbf{b} \in \mathbb{R}^m$. Let C be the convex cone generated by $\mathbf{a}_i$s. Exactly one of the following holds:

- $\mathbf{b} \in C$
- There exists a hyperplane H such that $\mathbf{0} \in H$ and H strictly separates $\mathbf{b}$ from C.
  That is $H = \{\mathbf{x} : \mathbf{h}^T\mathbf{x} = 0\}$ and $\forall i, \mathbf{h}^T\mathbf{a}_i \geq 0$ and $\mathbf{h}^T\mathbf{b} < 0$.



2: Prove Farkas lemma using separation theorem. (What does the separation give?)

Solution: From the separation theorem, there exists $\mathbf{h} \in \mathbb{R}^m$ and $z \in \mathbb{R}$ such that $\forall \mathbf{x} \in C, \mathbf{h}^T\mathbf{x} > z$ and $\mathbf{h}^T\mathbf{b} < z$. Since $\mathbf{0} \in C$, we get $\mathbf{h}^T\mathbf{0} = 0 > z$. We can try to replace z by 0 and get not strict separation for the cone.

What if $\exists \mathbf{x} \in C$ such that $\mathbf{h}^T\mathbf{x} < 0$? Then $1000\mathbf{x} \in C$ and $1000\mathbf{h}^T\mathbf{x} < z$ if 1000 big enough. Hence we can let $z = 0$.

Reformulations of Farkas lemma:

- $A\mathbf{x} = \mathbf{b}$ has a non-negative solution iff $\forall \mathbf{y} \in \mathbb{R}^m$ with $\mathbf{y}^T A \geq \mathbf{0}^T$ also $\mathbf{y}^T\mathbf{b} \geq 0$.
- $A\mathbf{x} \leq \mathbf{b}$ has a non-negative solution iff $\forall \mathbf{y} \in \mathbb{R}^m$, $\mathbf{y} \geq \mathbf{0}$ with $\mathbf{y}^T A \geq \mathbf{0}^T$ also satisfies $\mathbf{y}^T\mathbf{b} \geq \mathbf{0}$.
- $A\mathbf{x} \leq \mathbf{b}$ has a solution iff $\forall \mathbf{y} \in \mathbb{R}^m$, $\mathbf{y} \geq \mathbf{0}$ with $\mathbf{y}^T A = \mathbf{0}^T$ also satisfies $\mathbf{y}^T\mathbf{b} \geq \mathbf{0}$.

Lets have linear programs

$$\text{maximize } \mathbf{c}^T\mathbf{x} \text{ subject to } A\mathbf{x} \leq \mathbf{b} \text{ and } \mathbf{x} \geq \mathbf{0} \tag{P}$$

$$\text{minimize } \mathbf{b}^T\mathbf{y} \text{ subject to } A^T\mathbf{y} \geq \mathbf{c} \text{ and } \mathbf{y} \geq \mathbf{0} \tag{D}$$

Lemma (Weak Duality): Let $\mathbf{x}$ and $\mathbf{y}$ be feasible solutions of (P) and (D). Then

$$\mathbf{c}^T\mathbf{x} \leq \mathbf{b}^T\mathbf{y}.$$

3: Prove the weak duality.

Solution:

$$\mathbf{c}^T\mathbf{x} = \mathbf{x}^T\mathbf{c} \leq \mathbf{x}^T A^T\mathbf{y} = (A\mathbf{x})^T\mathbf{y} \leq \mathbf{b}^T\mathbf{y}$$

Proof of the Strong Duality theorem point 4. from worksheet 4 using the Farkas lemma.
The point 4 is saying ($\mathbf{c}^T\mathbf{x}^\star = \mathbf{b}^T\mathbf{y}^\star$.)

If both (P) and (D) have a feasible solution then each has an optimal solution, and if $\mathbf{x}^\star$ is an optimal solution of (P) and $\mathbf{y}^\star$ is an optimal solution of (D), then

$$\mathbf{c}^T\mathbf{x}^\star = \mathbf{b}^T\mathbf{y}^\star.$$

That is, the *maximum of* (P) *equals the minimum of* (D).

Let $\mathbf{x}^\star$ be optimal solution. Let $\gamma = \mathbf{c}^T\mathbf{x}^\star$.

4: Are there solutions to $A\mathbf{x} \leq \mathbf{b}$ and $\mathbf{c}^T\mathbf{x} \geq \gamma$?

Solution: Yes, $\mathbf{x}^\star$.

5: Are there solutions to $A\mathbf{x} \leq \mathbf{b}$ and $\mathbf{c}^T\mathbf{x} \geq \gamma + \varepsilon$, where $\varepsilon > 0$?

Solution: No, contradiction with $\mathbf{x}^\star$ being optimal.

Let $\hat{A} = \binom{A}{-\mathbf{c}^T}$ and $\hat{\mathbf{b}}_\varepsilon = \binom{\mathbf{b}}{-\gamma-\varepsilon}$.

6: Apply Farkas Lemma on $\hat{A}\mathbf{x} \le \hat{\mathbf{b}}_\varepsilon$ (which version?, write $\hat{\mathbf{y}}$ from FL as $(\mathbf{u}, z) \in \mathbb{R}^{m+1}$?)

Solution: FL: implies there exists $\hat{\mathbf{y}} \in \mathbb{R}^{m+1}$ such that $\hat{\mathbf{y}} \ge \mathbf{0}$, $\hat{\mathbf{y}}^T\hat{A} \ge \mathbf{0}^T$ and $\hat{\mathbf{y}}^T\hat{\mathbf{b}}_\varepsilon < 0$.

If we assign $(\mathbf{u}, z) = \hat{\mathbf{y}}$ we get

$$\mathbf{u}^T A - z \cdot \mathbf{c}^T \ge \mathbf{0}^T \text{ and } \mathbf{u}^T\mathbf{b} - z(\gamma + \varepsilon) < 0.$$

Which can be rewritten as

$$A^T\mathbf{u} \ge z \cdot \mathbf{c} \text{ and } \mathbf{u}^T\mathbf{b} < z(\gamma + \varepsilon).$$

Divide by z and we get

$$A^T\frac{\mathbf{u}}{z} \ge \mathbf{c} \text{ and } \frac{\mathbf{u}}{z}^T\mathbf{b} < (\gamma + \varepsilon).$$

Let $\mathbf{y}_\varepsilon = \frac{\mathbf{u}}{z}$. Then

$$\forall\varepsilon > 0, \exists\mathbf{y}_\varepsilon, A^T\mathbf{y}_\varepsilon \ge \mathbf{c} \text{ and } \mathbf{y}_\varepsilon^T\mathbf{b} < (\gamma + \varepsilon).$$

By taking limit for $\varepsilon \to 0$, we get that there exists $\mathbf{y}^\star$ such that $A^T\mathbf{y}^\star \ge \mathbf{c}$ and $\mathbf{b}^T\mathbf{y}^\star \le \gamma$. By weak duality $\mathbf{b}^T\mathbf{y}^\star = \gamma$ and $\mathbf{y}^\star$ is an optimal solution.

7: How to show that $z \neq 0$? (Hint: Use Farkas lemma again with $\varepsilon = 0$.)

Solution: Use Farkas Lemma with $\varepsilon = 0$. It changes to $\forall$. In particular, it gives that

$$\forall\hat{\mathbf{y}} \ge 0, \hat{\mathbf{y}}^T\hat{A} \ge \mathbf{0}^T \text{ and } \hat{\mathbf{y}}^T\hat{\mathbf{b}} \ge 0.$$

This means

$$\forall(\mathbf{u}, z) \ge 0, A^T\mathbf{u} \ge z \cdot \mathbf{c} \text{ and } \mathbf{u}^T\mathbf{b} \ge z\gamma$$

. and implies $\mathbf{u}^T\mathbf{b} \ge z\gamma$. If $z = 0$, we would get a contradiction with $\mathbf{u}^T\mathbf{b} < z(\gamma + \varepsilon)$.